Cell Decomposition and definable p-adic sets and functions

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Outline

1. Cell decomposition
   - semi-algebraic p-adic sets
2. Study of semi-affine p-adic sets
   - definition of semi-affine p-adic sets
   - cells
   - every definable set is a semi-affine set
3. definable functions
   - semi-affine case
   - a weaker language?

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What is cell decomposition?

- a cell is typically a set of the following type

$$\left\{ (x,t) \in \mathbb{Q}_p^{k+1} \;\middle|\; \begin{array}{l} x \in \text{ some definable set} \\ \text{t satisfies a condition of a fixed form} \end{array} \right\}$$

- purpose of cell decomposition: *partition* any definable set as a finite union of cells such that (for example) functions have a *simpler form* on each cell
- can for example be used to study the definable sets and functions in a given structure

($\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r.t. the norm $|x|_p = p^{-\mathrm{ord}\,(x)}$.)

Classical example: p-adic semi-algebraic sets

- Put $\mathcal{L}_{alg} := (+, -, \cdot, |, =, c \in \mathbb{Q}_p)$
- A subset of $\mathbb{Q}_p^k$ is semi-algebraic if it can be obtained by taking intersections, unions and complements (a finite number of times) of sets of the form

$$\{x \in \mathbb{Q}_p^k \mid \exists y \in \mathbb{Q}_p : f(x) = y^n\},$$

  where $f(x) \in \mathbb{Q}_p[x],\ x = (x_1, \ldots, x_k)$, and $n \in \mathbb{N}, n \geqslant 2$.
- Cell decomposition can be used to prove that the semi-algebraic subsets of $\mathbb{Q}_p^k$ are exactly the $\mathcal{L}_{alg}$-definable subsets of $\mathbb{Q}_p^k$.
  (See J.Denef, *p*-adic semi-algebraic sets and cell decomposition, 1986)

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- Put $\mathcal{L}_{aff} := (+, -, c \in \mathbb{Q}_p, \{\overline{c} \mid c \in \mathbb{Q}_p\}, |, \{Q_{n,m}\}_{n,m})$.

  In $\mathbb{Q}_p$, the interpretation of these symbols is as follows:
  - $\overline{c}$ denotes the multiplication map $\overline{c} : \mathbb{Q}_p \to \mathbb{Q}_p : x \mapsto cx$
  - by $x|y$ we mean $\operatorname{ord} x \leqslant \operatorname{ord} y$
  - $Q_{n,m} = \bigcup_{k\in\mathbb{Z}} p^{kn}(1 + p^m\mathbb{Z}_p)$

Definition (semi-affine set)

A semi-affine subset of $\mathbb{Q}_p^k$ is a set that can be obtained as a boolean combination (i.e. taking finite intersections, unions, complements) of sets of the following types:

$$\{x \in \mathbb{Q}_p^k \mid f_1(x) \in \lambda Q_{n,m}\} \quad \text{or} \quad \{x \in \mathbb{Q}_p^k \mid f_2(x) | f_3(x)\},$$

where the $f_i(x)$ are linear polynomials over $\mathbb{Q}_p$.

Definition (cell)

a cell in $\mathbb{Q}_p^k \times \mathbb{Q}_p$ with center $c(x)$ is a set of the form

$$\left\{ (x,t) \in D \times \mathbb{Q}_p \;\middle|\; \begin{array}{l} \operatorname{ord} a_1(x) \,\square_1\, \operatorname{ord} t - c(x) \,\square_2\, \operatorname{ord} a_2(x) \\ t - c(x) \in \lambda Q_{n,m} \end{array} \right\},$$

with

- $a_i(x), c(x)$ linear polynomials over $\mathbb{Q}_p$
- D a semi-affine subset of $\mathbb{Q}_p^k$
- $\square_i$ denotes $<$ or ‘no condition".

Goal: use cell-decomposition to prove that the semi-affine subsets of $\mathbb{Q}_p^k$ are the $\mathcal{L}_{aff}$-definable subsets of $\mathbb{Q}_p^k$ (joined work with R. Cluckers)

intersection of cells

Theorem (Intersection of cells)

Let A_1, A_2 be cells with centers $c_1(x)$, resp. $c_2(x)$. Then $A_1 \cap A_2$ can be written as a finite union of disjoint cells A with center either $c_1(x)$ or $c_2(x)$, and such that on each cell A there exists $\lambda_1, \lambda_2, \lambda_3 \in \Lambda_{n,m}$, such that

$$\begin{aligned} t - c_1(x) &\in \lambda_1 Q_{n,m}, \\ t - c_2(x) &\in \lambda_2 Q_{n,m}, \\ c_1(x) - c_2(x) &\in \lambda_3 Q_{n,m}, \end{aligned}$$

for all $(x, t) \in A$.

Corollary: any boolean combination of cells can be written as a finite union of cells

Cell Decomposition theorem

Theorem (Cell Decomposition)

Let t be one variable and $x = (x_1, \ldots, x_m)$. Let $f_1(x,t), \ldots, f_r(x,t)$ be linear polynomials over $\mathbb{Q}_p$. There exists a finite partition of $\mathbb{Q}_p^k \times \mathbb{Q}_p$ into cells A, such that each cell A has a center $c(x)$ such that for all $(x,t) \in A$, $f_i(x,t)$ has one of the following forms:

$$f_i(x,t) = u_i^{(m)}(x,t)h_i(x),$$

or

$$f_i(x,t) = u_i^{(m)}(x,t)a_i(t - c(x))$$

where $u_i^{(m)}(x,t) \in 1 + p^m\mathbb{Z}_p$,
$h_i(x)$ is a linear polynomial over $\mathbb{Q}_p$, and $a_i \in \mathbb{Q}_p$.

Sketch of proof

Theorem (decomposition of semi-affine sets)

Every semi-affine set can be written as a finite union of cells

- It suffices to prove that $\{x \in \mathbb{Q}_p^k \mid f_1(x) \in \lambda Q_{n,m}\}$ and $\{x \in \mathbb{Q}_p^k \mid f_2(x) | f_3(x)\}$ (and complements) can be written as a finite union of cells
- For $\{x \in \mathbb{Q}_p^k \mid f_2(x) | f_3(x)\}$ this follows from the previous theorem.

Theorem (semi-affine sets are closed under projection)

Let S be a semi-affine subset of $\mathbb{Q}_p^{k+l}$, then the set $\{x \in \mathbb{Q}_p^k \mid \exists t \in \mathbb{Q}_p^l : (x,t) \in S\}$ is a semi-affine set.

(Proof simplifies to $\{x \in \mathbb{Q}_p^k \mid \exists t \in \mathbb{Q}_p : (x,t) \in C\}$, for a cell C.)

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semi-affine functions

Definition

A function $f : A \subseteq \mathbb{Q}_p^k \to \mathbb{Q}_p^l$ is semi-affine if its graph is a semi-affine set.

A semi-affine function can be described as follows:

Lemma

Let $f : A \subseteq \mathbb{Q}_p^k \to \mathbb{Q}_p^l$ be a semi-affine function. There exists a finite partition of A in cells C, such that on each cell C, f has the form

$$f_{|C} : C \to \mathbb{Q}_p^l : x \mapsto (f_1(x), \ldots, f_l(x)),$$

where $f_i(x)$ is a linear polynomial over $\mathbb{Q}_p$.

semi-affine vs semi-algebraic

The multiplication map $\cdot : \mathbb{Q}_p^2 \to \mathbb{Q}_p : (x, y) \mapsto xy$
is not a semi-affine function

BUT

The semi-affine subsets $\subseteq \mathbb{Q}_p$ coincide with the semi-algebraic subsets of $\subseteq \mathbb{Q}_p$

Question

Is it possible to find a weaker language

- that conserves the definable subsets $\subseteq \mathbb{Q}_p$,
- such that $+ : \mathbb{Q}_p^2 \to \mathbb{Q}_p : (x, y) \mapsto x + y$ is no longer definable?

Answer: YES !!

- Put $\mathcal{L}_{min} := (\{C_n\}_n, \{R_{n,m}\}_{n,m}, c \in \mathbb{Q}_p)$.

  In $\mathbb{Q}_p$, the interpretation of these symbols is as follows:
  - $C_n(x, y; z, t)$ is true if $n + \mathrm{ord}\,(x - y) < \mathrm{ord}\,(z - t)$
  - $R_{n,m}(x, y, z)$ is true if $y \in x + zQ_{n,m}$
- For the $\mathcal{L}_{min}$-structure $\mathbb{Q}_p$, we can prove
  - Cell decomposition
  - Elimination of Quantifiers

This language answers the posed question, since

- semi-affine sets $\subseteq \mathbb{Q}_p$ are definable with these symbols
- After a finite partition in cells, every $\mathcal{L}_{min}$-definable function has the form $f_{|C} : C \to \mathbb{Q}_p^l : x \mapsto (f_1(x), \ldots, f_l(x))$, with $f_i(x) \in \{x_1, \ldots, x_k\}$, or $f_i(x) = a_i \in \mathbb{Q}_p$